

CHAPTER 17 - FOURIER TRANSFORM

List of topics for this chapter :

- Fourier Transform and its Properties
- Circuit Applications
- Parseval's Theorem
- Applications

FOURIER TRANSFORM AND ITS PROPERTIES

Problem 17.1 Find the Fourier Transform of the pulse shown in Figure 17.1.

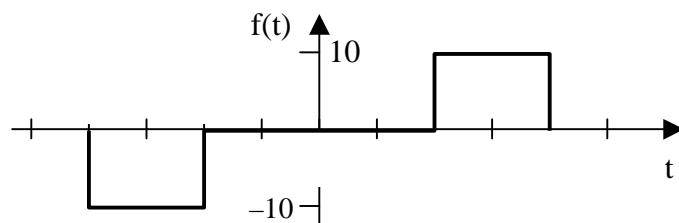


Figure 17.1

We begin with the derivative of $f(t)$.

$$\frac{df(t)}{dt} = -\delta(t+2) + \delta(t+1) + \delta(t-1) - \delta(t-2)$$

Transforming this into the frequency domain yields,

$$j\omega F(\omega) = -e^{j2\omega} + e^{j\omega} + e^{-j\omega} - e^{-j2\omega} = 2\cos(2\omega) - 2\cos(\omega)$$

Therefore,

$$F(\omega) = \frac{2(\cos(2\omega) - \cos(\omega))}{j\omega}$$

Problem 17.2 Find the inverse Fourier transforms of the following,

- (a) $10/[(j\omega)(j\omega + 5)]$
- (b) $5j\omega/[-j\omega + 1)(j\omega + 2)]$
- (c) $(2 - j\omega)/(-\omega^2 + 4j\omega + 3)$
- (d) $3\delta(\omega)/[(j\omega + 2)(j\omega + 3)]$

Now to find the inverse transforms.

$$(a) \quad F(s) = \frac{10}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5}, \quad A = 10/5 = 2 \text{ and } B = 10/-5 = -2$$

$$\text{Therefore, } F(\omega) = (2/j\omega) - (2/(j\omega + 5))$$

$$\text{Transforming, } f(t) = \underline{\text{sgn}(t) - 2e^{-5t}u(t)}$$

$$(b) \quad F(s) = \frac{5s}{(-s+1)(s+2)} = \frac{-5s}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$A = -5/(1+2) = -5/3 \text{ and } B = -5 \times (-2)/(-2-1) = -10/3$$

$$\text{Therefore, } F(\omega) = [(-5/3)/(j\omega - 1)] + [(-10/3)/(j\omega + 2)]$$

$$\text{Transforming, } f(t) = \underline{-\frac{5}{3}e^t u(-t) - \frac{10}{3}e^{-2t} u(t)}$$

$$(c) \quad F(s) = \frac{(2-s)}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}, \quad A = 3/2 \text{ and } B = -5/2$$

$$\text{Therefore, } F(\omega) = [1.5/(j\omega + 1)] - [2.5/(j\omega + 3)]$$

$$\text{Transforming, } f(t) = \underline{1.5e^{-t}u(t) - 2.5e^{-3t}u(t)}$$

$$(d) \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{3\delta(\omega)e^{j\omega t}}{(j\omega+2)(j\omega+3)} d\omega = \frac{1}{2\pi} \frac{3}{6} = \underline{\frac{1}{4\pi}}$$

Problem 17.3 [17.7] Find the Fourier transform of the "sine-wave pulse" shown in Figure 17.1.

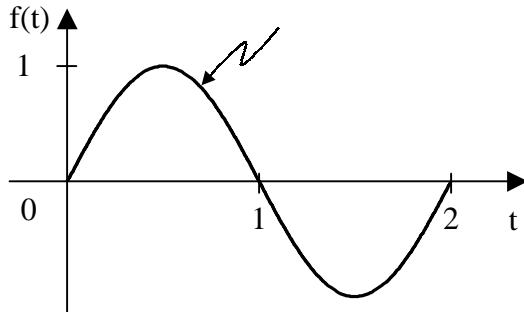


Figure 17.1

$$f(t) = \sin(\pi t)[u(t) - u(t-2)]$$

$$F(\omega) = \int_0^2 \sin(\pi t) e^{-j\omega t} dt = \frac{1}{2j} \int_0^2 (e^{j\pi} - e^{-j\pi})(e^{-j\omega t}) dt$$

$$F(\omega) = \frac{1}{2j} \int_0^2 e^{j(-\omega+\pi)t} + e^{-j(\omega+\pi)t} dt$$

$$F(\omega) = \frac{1}{2j} \left[\frac{1}{-j(\omega-\pi)} e^{-j(\omega-\pi)t} \Big|_0^2 + \frac{1}{-j(\omega+\pi)} e^{-j(\omega+\pi)t} \Big|_0^2 \right]$$

$$F(\omega) = \frac{1}{2} \left(\frac{1 - e^{-j2\omega}}{\pi - \omega} + \frac{1 - e^{-j2\omega}}{\pi + \omega} \right)$$

$$F(\omega) = \frac{1}{(2)(\pi^2 - \omega^2)} (2\pi + 2\pi e^{-j2\omega})$$

$$F(\omega) = \frac{\pi}{\omega^2 - \pi^2} (e^{-j2\omega} - 1)$$



CIRCUIT APPLICATIONS

Problem 17.4 Find the transfer function, $V_o(\omega)/V_s(\omega)$ for the circuit shown in Figure 17.3.

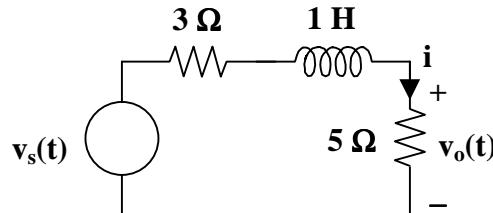


Figure 17.1

First we will solve for I .

$$I = \frac{V_s(\omega)}{3 + j\omega + 5} = \frac{V_s(\omega)}{j\omega + 8}, \text{ and } V_o(\omega) = 5I$$

Therefore,

$$\frac{V_o(\omega)}{V_s(\omega)} = \frac{5}{j\omega + 8}$$

Problem 17.5 Solve for $v_C(t)$ in Figure 17.4, where $i(t) = u(t)$ A.

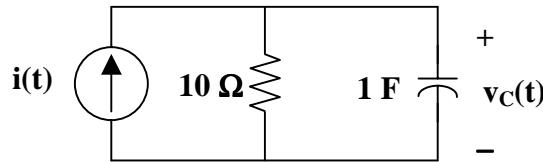


Figure 17.1

First we transform $i(t)$ into the frequency domain.

$$I(\omega) = \pi\delta(\omega) + 1/(j\omega), \text{ and } V_C(\omega) = I(\omega) \frac{1}{j\omega} = I(\omega) \frac{1}{j\omega + 0.1}$$

Therefore, $V_C(\omega) = \frac{\pi\delta(\omega)}{j\omega} + \frac{1}{j\omega(j\omega + 0.1)} = V_1 + V_2$

$$V_2 = \frac{1}{s(s + 0.1)} = \frac{A}{s} + \frac{B}{s + 0.1}, \text{ where } A = 1/0.1 = 10 \text{ and } B = 1/(-0.1) = -10$$

Therefore, $v_2(t) = 5\text{sgn}(t) - 10e^{-t/10}u(t)$

$$v_1(t) = \frac{1}{2\pi} \int \frac{\pi\delta(\omega)}{j\omega + 0.1} e^{j\omega t} d\omega = \frac{1}{2\pi} \frac{\pi}{0.1} = 5$$

This leads to $v_o(t) = 5 - 5\text{sgn}(t) - 10e^{-t/10}u(t)$, but $\text{sgn}(t) = -1 + 2u(t)$

Therefore, $v_o(t) = 5 - 5 + 10u(t) - 10e^{-t/10}u(t)$

or $v_o(t) = \underline{10(1 - e^{-t/10})u(t)}$ volts

Problem 17.6 [17.29] Determine the current $i(t)$ in the circuit of Figure 17.1(b), given the voltage source shown in Figure 17.1(a).

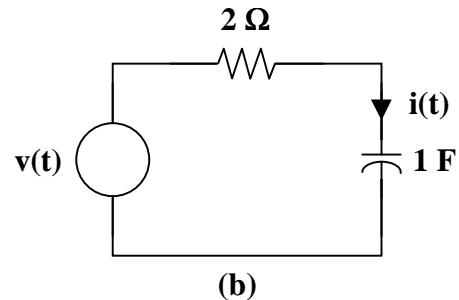
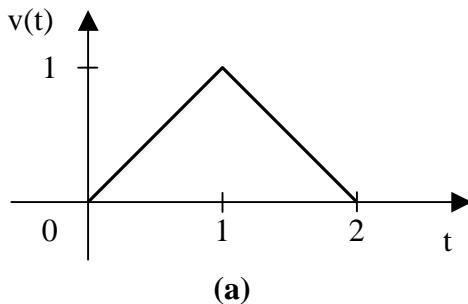


Figure 17.1

$$\dot{v}(t) = \delta(t) - 2\delta(t-1) + \delta(t-2)$$

$$-\omega^2 V(\omega) = 1 - 2e^{-j\omega} + e^{j\omega 2}$$

$$V(\omega) = \frac{1 - 2e^{-j\omega} + e^{j\omega 2}}{-\omega^2}$$

Now, $Z(\omega) = 2 + \frac{1}{j\omega} = \frac{1 + j2\omega}{j\omega}$

$$I = \frac{V(\omega)}{Z(\omega)} = \frac{2e^{j\omega} - e^{j\omega 2} - 1}{\omega^2} \cdot \frac{j\omega}{1 + j2\omega}$$

$$I = \frac{1}{(j\omega)(0.5 + j\omega)} (0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

But $\frac{1}{(s)(s+0.5)} = \frac{A}{s} + \frac{B}{s+0.5} \longrightarrow A = 2, B = -2$

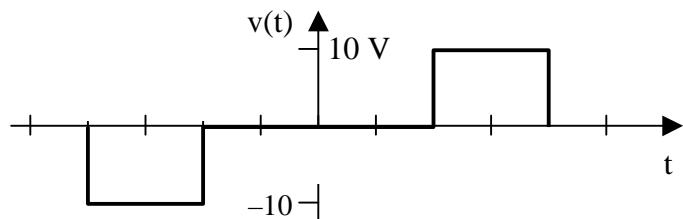
$$I(\omega) = \left(\frac{2}{j\omega} \right) (0.5 + 0.5e^{j\omega 2} - e^{-j\omega}) - \left(\frac{2}{0.5 + j\omega} \right) (0.5 + 0.5e^{-j\omega 2} - e^{-j\omega})$$

$$i(t) = \frac{1}{2} \text{sgn}(t) + \frac{1}{2} \text{sgn}(t-2) - \text{sgn}(t-1) - e^{-0.5t} u(t) - e^{-0.5(t-2)} u(t-2) - 2e^{-0.5(t-1)} u(t-1)$$



PARSEVAL'S THEOREM

Problem 17.7 Find the total energy in $v(t)$ where $v(t)$ is the pulse shown below.



In the time domain, $W_{1\Omega} = \int_{-2}^{-1} (-10)^2 dt + \int_1^2 10^2 dt = 100t \Big|_{-2}^{-1} + 100t \Big|_1^2$

$$= -100 + 200 + 200 - 100 = \underline{\underline{200 \text{ J}}}$$

Problem 17.8 [17.43] A voltage source $v_s(t) = e^{-t} \sin(2t) u(t)$ V is applied to a $1\text{-}\Omega$ resistor. Calculate the energy delivered to the resistor.

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t) dt = \int_0^{\infty} e^{-2t} \sin^2(2t) dt$$

But $\sin^2(A) = \frac{1}{2} [1 - \cos(2A)]$

$$W_{1\Omega} = \int_0^{\infty} e^{-2t} (0.5)[1 - \cos(4t)] dt = \frac{1}{2} \cdot \frac{e^{-2t}}{-2} \Big|_0^{\infty} - \frac{e^{-2t}}{4+16} [-2\cos(4t) + 4\sin(4t)] \Big|_0^{\infty}$$
$$W_{1\Omega} = \left(\frac{1}{4}\right) + \left(\frac{1}{20}\right) = \underline{\underline{0.15 \text{ J}}}$$



APPLICATIONS

Problem 17.9 Given the AM signal,

$$f(t) = 10(1 + 4\cos(2000\pi t))\cos(\pi \times 10^6 t),$$

solve for the:

- (a) the carrier frequency
- (b) the lower sideband frequency
- (c) the upper sideband frequency

$$\omega_m = 2000\pi = 2\pi f \text{ which leads to } f = 1 \text{ kHz}$$

- (a) $\omega_c = \pi \times 10^6 = 2\pi f_c$ which leads to $f_c = \underline{\underline{500 \text{ kHz or } 0.5 \text{ MHz}}}$
- (b) $L_{\text{sb}} = f_c - f_m = (500 - 1) \text{ kHz} = 499 \text{ kHz}$
- (c) $U_{\text{sb}} = f_c + f_m = (500 + 1) \text{ kHz} = 501 \text{ kHz}$

Problem 17.10 [17.47] A voice signal occupying the frequency band of 0.4 to 3.5 kHz is used to amplitude modulate a 10-MHz carrier. Determine the range of frequencies for the lower and upper sidebands.

For the lower sideband, the frequencies range from

$$10,000,000 - 3,500 = \underline{\underline{\mathbf{9,996,500 \, Hz}}}$$

to

$$10,000,000 - 400 = \underline{\underline{\mathbf{9,999,600 \, Hz}}}$$

For the upper sideband, the frequencies range from

$$10,000,000 + 400 = \underline{\underline{\mathbf{10,000,400 \, Hz}}}$$

to

$$10,000,000 + 3,500 = \underline{\underline{\mathbf{10,003,500 \, Hz}}}$$